NASA TECHNICAL TRANSLATION

NASA TT F-195

NASA TT F-195

ON B. G. GALERKIN'S METHOD FOR THE SOLUTION OF BOUNDARY VALUE PROBLEMS

by M. V. Keldysh

From *Izvestiya Akademii Nauk SSSR, Seriya Matematicheskaya*
Vol. 6, No. 4, 1942

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MARCH 1964

ON B. G. GALERKIN'S METHOD FOR THE SOLUTION OF BOUNDARY VALUE PROBLEMS

By M. V. Keldysh

Translation of "O metode B. G. Galerkina dlya resheniya krayevykh zadach"

From Izvestiya Akademii Nauk SSSR, Seriya Matematicheskaya
Vol. 6, No. 4, pp. 309-330, 1942

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Office of Technical Services, Department of Commerce,
Washington, D.C. -- Price $1.50

ON B. G. GALERKIN'S METHOD FOR THE SOLUTION OF BOUNDARY VALUE PROBLEMS

by

M. V. Keldysh

SUMMARY

In 1915 B. G. Galerkin proposed a method for the solution of boundary value problems of ordinary differential equations. As became known later, in the case of variational problems this method coincides essentially with Rietz's method. But Galerkin's method does not depend on the variational nature of problems and may therefore be applied to non-selfadjoint equations.

In the present paper we prove the applicability of the method to some classes of differential equations.

In Section 1 we consider the general ordinary differential equation of order 2n. Section 2 contains a detailed investigation of all boundary conditions of the second-order equation, which relate to the values of the functions and of the derivative at each endpoint of the interval. In Section 3 we consider the Dirichlet problem for partial differential equations. The convergence (in the mean) of the solutions and the convergence of the eigenvalues are established in the general case.

This article provides a proof of the convergence of Galerkin's method for certain types of linear differential equations.

In 1915 B. G. Galerkin proposed a new method for the solution of boundary value problems of ordinary differential equations, which he applied to the solution of a number of stability problems in elasticity theory[(1)]. Later it became clear that in the case of variational problems this method is essentially the same as that of Rietz. However, the application of Galerkin's method does not depend on the variational problem which determines the differential equations, and can be also applied to non-selfadjoint equations. Recently Galerkin's method was widely applied to non-selfadjoint systems in studies of non-conservative mechanical systems, and invariably yielded good results.

Rietz's method for the solution of variational problems was substantiated for the simplest cases in the works of Rietz himself. After the work of Rietz, a number of studies were devoted to this method, and, particularly, it was studied in depth in the fundamental works of N. M. Krylov and N. N. Bogolyubov. On the other hand, insofar as we know, the application of Galerkin's method to non-selfadjoint systems has not yet been substantiated. In certain quarters[(2,3)] doubts were even expressed about the validity of its application. Recently G. I. Petrov[(4)] published a paper in which he gave a theoretical justification of Galerkin's method for a number of special cases, by reducing the convergence problem to the investigation of an infinite system of linear equations.

We shall also take advantage of the connection between Galerkin's method and systems of linear equations, and we will prove the convergence of this method in a great number of cases by imposing a number of necessary restrictions on the set of approximating functions.

In Section 1 we consider the general ordinary differential equation of order 2n, and we confine ourselves to the simplest boundary conditions. The carrying over of the proof to other boundary conditions should present no major difficulties.

In Section 2 we analyze in detail all boundary conditions of the second-order equation which relate the values of the functions and of the derivative at each endpoint of the interval.

In Section 3 we consider the Dirichlet problem for partial differential equations. In this case, under the general restrictions imposed on the set of functions, we no longer have uniform convergence of the solutions. We establish, however, their convergence in the mean, and the convergence of the eigenvalues. In this case, the investigation of the infinite system is also considerably more difficult.

SECTION 1. THE EQUATION OF ORDER 2n

Let us consider on the interval $0 \leq x \leq 1$ the equation of order 2n

$$L(y) = p(x) \frac{d^{2n}y}{dx^{2n}} + \sum_{i=0}^{2n-1} p\ (x, \lambda) \frac{d^i y}{dx^i} = f(x), \tag{1}$$

depending on the parameter λ which varies in the region D of the complex plane. The coefficients of the equation are assumed to be continuous functions of x and λ, when x varies on the interval (0.1) and λ in the region D, and analytic functions of λ in D. We will assume that the coefficients are differentiable with respect to x as many times as will be necessary, and that

$$p(x) > 0.$$

We will consider the system S formed by equation (1) and the simplest boundary conditions

$$y^{(k)}(0) = y^{(k)}(1), \quad k = 0,1,\dots, n-1 \qquad (2)$$

For a fixed λ, either the system S, has, as is known, a solution for any value of the right-hand side of (1), or the homogeneous system S (obtained by set up f (x) = 0) admits a solution different from zero. The values λ for which the latter condition holds are known as eigenvalues.

Academician B. G. Galerkin proposed a method for solving the system S, which required the solution of a system of linear algebraic equations for the construction of each approximation. This method consists of the following:

Let

$$\varphi_1(x), \quad \varphi_2(x), \dots, \varphi_m(x), \dots \qquad (3)$$

be a set of functions satisfying the boundary conditions (2). Let

$$y_m = x_1^{(m)}\varphi_1(x) + x_2^{(m)}\varphi_2(x) + \cdots + x_m^{(m)}\varphi_m(x) \tag{4}$$

and to determine the constants $x_k^{(m)}$ construct a system of linear algebraic equations as follows:

$$\int_0^1 [L(y_m) - f]\varphi_i dx = 0, \qquad i = 1, 2, \ldots, m. \tag{5}$$

This system has the following form:

$$\sum_{k=1}^{m} c_{ik}(\lambda)\, x_k^{(m)} - f_i = 0, \tag{6}$$

and its coefficients are holomorphic functions of λ in D.

The eigenvalues of the system S will be sought as the limits of the eigenvalues of the algebraic system (6), and if λ is not an eigenvalue of the system S, its solution will be sought as the limit of the sequence

$$y_1(x),\ y_2(x),\dots,y_m(x),\dots \tag{7}$$

with coefficients determined from the equations (6).

As is known, in the case that equation (4) can be obtained as Euler's equation for a real functional, the method of Academician Galerkin coincides with the method of Rietz for the solution of the corresponding variational problem. As distinct from Rietz's method, however, the method of Academician Galerkin can be applied also to non-selfadjoint equations, and to equations with complex coefficients.

We shall prove the following proposition:

If the set of functions consisting of the first n powers of x and of the n-th derivatives of the functions of sequence (3)

$$1,\ x,\ x^2,\dots,\ x^{n-1},\ \varphi_1^{(m)},\ \varphi_2^{(n)},\dots,\varphi_m^{(n)}\dots \tag{8}$$

is complete in the mean square deviation sense, then

1. The eigenvalues of the system S are obtained by a limiting process from the eigenvalues of system (6).

2. The eigenfunctions of the system S are obtained as

limits of the sequence (7), whose coefficients are obtained from the solution of the homogeneous equations (6), which correspond to the eigenvalues of system (6).

3. If λ is not an eigenvalue of the system S, the solution of S is the limit of the sequence (7) with coefficients determined from (6) for the same value of λ.

For $j < n$ the derivatives $y_m^{(j)}(x)$ converge uniformly to the corresponding derivatives of the solution y, while $y_m^{(n)}(x)$ converges in the mean to $y^{(n)}(x)$.

For purposes of proof we note that equation (1) can always be written in the form

$$\frac{d^n}{dx^n}(py^{(n)}) + \frac{d^n}{dx^n}\sum_{j=0}^{n-1} q_j(x,\lambda)y^{(j)} + \sum_{j=0}^{n-1} r_j(x,\lambda)y^{(j)} = f\ (x). \qquad (1')$$

Further, in the proof we can assume that the functions $\varphi_m^{(n)}(x)$ form an orthonormal set with weight p (x)

$$\int_0^1 p\varphi_m^{(n)}\varphi_k^{(n)}dx = \begin{cases} 0, & m \neq k; \\ 1, & m = k. \end{cases} \qquad (9)$$

In fact, if this is not the case, we can always construct a new set of functions $\psi_m(x)$, obtained from the set (3) by a linear transformation which orthogonalizes the set of n-th derivatives (3)

$$\psi_m = \sum_{i=0}^{m} \alpha_{mk}\,\varphi_k, \qquad \varphi_m = \sum_{i=0}^{m} \beta_{mk}\,\psi_k.$$

The set of algebraic equations of the m-th approximation which is obtained initially from the sequence of functions ψ_m, can be excluded from system (6) by the change of variables

$$y_i^{(m)} = \sum_{k=i}^{m} \beta_{ki}\, x_k^{(m)}$$

and by the formation of linear combinations of the equations of system (6)

$$\bar{\Lambda}_i = \sum_{k=1}^{i} \alpha_{ik}\,\Lambda_k.$$

It follows that the eigenvalues of system (6) and $\bar{\Lambda}_i = 0$ coincide, and that the sequence (7) remains unchanged in the transition from the sequence φ_m to the sequence ψ_m.

Let us note several properties of the expansions with respect to the set of functions $\varphi_m^{(n)}(x)$.

By virtue of the boundary conditions (2), every function $\varphi_m^{(n)}(x)$ is orthogonal to all powers x^k, for $k < n$,

$$\int_0^1 x^k \varphi_m^{(n)}(x)\, dx = 0, \qquad k < n.$$

From this, and from the completeness of the set of functions (8) it follows that any function $f(x)$ which is orthogonal to all $\varphi_m^{(n)}$

$$\int_0^1 f\varphi_m^{(n)}\, dx = 0,$$

is a polynomial of degree n-1

$$f(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1},$$

and any function g (x) which is orthogonal to the first powers of x

$$\int_0^1 g x^k \, dx = 0, \qquad k < n,$$

can be approximated in the mean by linear combinations of the functions $\varphi_m^{(n)}$. In particular, if the set $\varphi_m^{(n)}$ is orthogonalized with weight p (x), the Fourier expansion of g (x) with respect to the functions $\varphi_m^{(n)}$ converges in the mean to g (x).

If the functions $\varphi_m^{(n)}(x)$ are orthogonal, then by virtue of the boundary conditions,

$$\int_0^1 \varphi_k \frac{d^n}{dx^n} (p\varphi_m^{(n)}) dx = (-1)^n \int_0^1 p\varphi_k^{(n)} \varphi_m^{(n)} dx = \begin{cases} (-1)^n, & m = k; \\ 0, & m \neq k. \end{cases}$$

Consequently, system (6) can be written in the form

$$x_i^{(m)} + \sum_{k=1}^{m} A_{ik}(\lambda)\, x_k^{(m)} - f_i = 0, \tag{10}$$

where

$$\left.\begin{aligned}
(-1)^n A_{ik} &= \int_0^1 \Big[\sum_{j=0}^{n-1} \frac{d^n}{dx^n}\left(q_j \varphi_k^{(j)}\right) + \sum_{j=0}^{n-1} r_j \varphi_k^{(j)}\Big] \varphi_i\, dx = 0,\\
(-1)^n f_i &= \int_0^1 f \cdot \varphi_i\, dx.
\end{aligned}\right\} \tag{11}$$

Further, we put

$$A_{ik} = a_{ik} + b_{ik},$$

where

$$\left.\begin{aligned}
a_{ik} &= \int_0^1 \varphi_i^{(n)} \sum_{j=0}^{n-1} q_j \varphi_k^{(j)}\, dx,\\
b_{ik} &= (-1)^n \int_0^1 \varphi_i \sum_{j=0}^{n-1} r_j \varphi_k^{(j)}\, dx.
\end{aligned}\right\} \tag{12}$$

The expression a_{ik} is obtained by integrating by parts n times the term A_{ik} which corresponds to the first term in the square bracket.

The system (10) is obtained by truncating the infinite system of linear equations

$$x_i + \sum_{k=1}^{\infty} A_{ik}(\lambda)\, x_k - f_i = 0. \tag{13}$$

We shall prove that the system (13) is equivalent to the system S in the following sense: To the solution y (x) of the system S there corresponds the solution of system (13)

$$x_k = \int_0^1 p y^{(n)} \varphi_k^{(n)}\, dx \tag{14}$$

with convergent sum of squares

$$\sum_{k=1}^{\infty} |x_k^2| < +\infty,$$

and conversely, to any solution of system (13) with convergent sum of squares there corresponds a solution of the system S. In particular, the eigenvalues of systems (13) and S coincide.

Let y (x) be the solution of system S. We shall expand $y^{(n)}$ in a Fourier series with respect to the orthogonal set of functions $\varphi_m^{(n)}(x)$. Determining the coefficients according to formulas (14), we obtain the expansion

$$y^{(n)}(x) \approx \sum_{m=1}^{\infty} x_m \varphi_m^{(n)}(x), \tag{15}$$

which converges in the mean, since $y^{(n)}(x)$ is orthogonal to

$$1, x, x^2, \dots , x^{n-1}$$

by virtue of the boundary conditions. Integrating this expansion, and determining the constants from the boundary conditions, we obtain the uniformly converging expansions

$$y^{(j)} (x) = \sum_{m=1}^{\infty} x_m \varphi_m^{(j)}, \qquad j < m. \tag{15'}$$

Multiplying equation (1') by $\varphi_i(x)$ and integrating on the interval (0, 1) we have, after integration by parts, the following relation:

$$\int_0^1 \left[py^{(n)}\varphi_i^{(n)} + \sum_{j=0}^{n-1} q_j y^{(j)}\varphi_i^{(n)} + (-1)^n \sum_{j-0}^{n-1} p_j y^{(j)}\varphi_i - (-1)^n f\varphi_i\right] dx = 0. \tag{16}$$

Substituting in the above the expansions (15) and (15') and integrating the series term by term, we ascertain that the x_k satisfy system (13).

Conversely, let us assume that x_k is a solution of system (13) with convergent sum of squares. Applying the Fisher-Riess theorem, we form the function $\eta\ (x)$ with the series expansion

$$\eta\ (x) \approx \sum_{m=1}^{\infty} x_m \varphi_m^{(n)}\ (x) \tag{17}$$

with respect to the orthogonal system $\varphi_m^{(n)}$, and we put

$$y\ (x) = \sum_{m=1} x_m \varphi_m\ (x).$$

The series thus obtained and its derivatives of order $< n$ converge uniformly, since they are obtained by integrating the series (17), and in addition

$$y^{(n)}(x) = \eta(x)$$

almost everywhere. The function $y(x)$ clearly satisfies the boundary conditions, and it must be shown that it satisfies the differential equation (1'). It follows from the definition of $y(x)$ that the equations (13) can be written in the form (16). Integrating by parts n times the last two terms of the integral expression (16), we obtain

$$\int_0^1 \left[py^{(n)} + \sum_{j=0}^{n-1} q_j y^{(j)} + \int_0^x \cdots \int_0^x \left(\sum_{j=0}^{n-1} r_j y^{(j)} - f \right) dx^n \right] \varphi_i^{(n)} dx = 0.$$

It follows that the function in the square brackets, being orthogonal to all $\varphi_m^{(n)}$, will be a polynomial of degree $n - 1$

$$py^{(n)} + \sum_{j=0}^{n-1} q_j y^{(j)} + \int_0^x \cdots \int_0^x \sum_{j=0}^{n-1} r_j y^{(j)} dx^n = C_0 + C_1 x + \cdots + C_n x^n.$$

Differentiating this relation n times, we see that y (x) satisfies equation (1).

We shall now prove that the eigenvalues of system (13) are obtained by a limiting process from the eigenvalues of system (10), and the solutions $x_1, \ldots, x_m, \ldots$ of system (13) are obtained by a limiting process from the solutions $x_1^{(m)}, \ldots, x_m^{(m)}$ of system (10), and that strong convergence holds

$$\lim_{m \to x} \sum_{k=1}^{\infty} |x_k - x_k^{(m)}|^2 = 0. \tag{18}$$

By virtue of Koch's results for linear systems it suffices to show that the series

$$\sum_{i,k} |A_{ik} (\lambda)|^2 \tag{19}$$

converges uniformly in the region $D^{(1)}$. From (18) it follows immediately that the sequence $y_m^{(n)}$ converges in the mean to the n-th derivative of the solution of S, and $y_m^{(j)}$ converges uniformly to $y^j(x)$ for $j < n$.

We will prove that the series (19) converges in the interior of the region D in which λ varies, and that the sum of the series is uniformly bounded in the interior of D.

Since the terms of the series (19) are the squares of the moduli of analytic functions, it follows that the series (19) converges uniformly in the interior of D*.

It suffices to show the convergence and boundedness of the sum of the series

$$\sum_{i,k} |a_{ik}|^2, \qquad \sum_{i,k} |b_{ik}|^2.$$

Let us consider the first series. By virtue of Bessel's inequality, we have for the orthogonal set $\varphi_m^{(n)}$ with weight p (x)

$$\sum_{i=1}^{\infty} |a_{ik}|^2 \le \int_0^1 \left| \sum_{j=0}^{n-1} q_j \varphi_k^{(j)} \ (x) \right|^2 \frac{dx}{p(x)},$$

since a_{ik} is the i-th Fourier coefficient of the function

*The latter follows from the fact that in any subregion D, the series (19) is dominated by a series of harmonic functions with bounded sum, and from Harnak's theorem.

$$\frac{1}{p(x)} \sum_{j=0}^{n-1} q_j \varphi_k^{(j)}.$$

Thus

$$\sum_{i,k} |a_{ik}|^2 \leq \sum_{k=1}^{\infty} \int_0^1 \left| \sum_{j=0}^{n-1} q_j \varphi^{(j)}(x) \right|^2 \frac{dx}{p(x)} . \qquad (20)$$

Expressing $\varphi_k^{(j)}(x)$ in terms of $\varphi_k^{(n)}(x)$ by the formula

$$\varphi_k^{(j)}(x) = \frac{1}{(n - j - 1)!} \int_0^x (x - t)^{n-j-1} \varphi_k^{(n)}(t)\, dt,$$

we obtain

$$\sum_{j=0}^{n-1} q_j \varphi_k^{(j)}(x) = \int_0^x \left(\sum_{j=0}^{n-1} q_j(x) \frac{(x - 1)^{n-j-1}}{(n - j - 1)!} \right) \varphi_k^{(n)}(t)\, dt$$

Consequently, the left side of this equality is the Fourier coefficient of a function of t which is obtained by dividing the bracket in the integrand into p (t) on the interval $0 \le t \le x$ and zero on $x < t \le 1$.

By virtue of Bessel's inequality

$$\sum_{k=1}^{\infty} \left| \sum_{j=0}^{n-1} q_j \varphi_k^{(j)} \right|^2 \le \int_0^x \left| \sum_{j=0}^{n-1} q_j(x,\lambda) \frac{(x-t)^{n-j-1}}{(n-j-1)!} \right|^2 \frac{dt}{p(t)}$$

and by virtue of (20)

$$\sum_{i,k} |a_{ik}|^2 \le \int_0^1 \frac{dx}{p(x)} \int_0^x \left| \sum_{j=0}^{n-1} q_j(x) \frac{(x-t)^{n-j-1}}{(n-j-1)!} \right|^2 \frac{dt}{p(t)} .$$

It follows that the series $\sum |a_{ik}|^2$ converges and that its sum is bounded in the interior of the region D.

Let us now consider the series $\sum |b_{ik}(\lambda)|^2$. Using the expression for b_{ik}, we have, by Schwartz's inequality

$$|b_{ik}|^2 \le \int_0^1 |\varphi_j|^2 dx \cdot \int_0^1 \left| \sum_{j=0}^{n-1} r_j \varphi_k^{(j)} \right|^2 dx$$

whence

$$\sum_{i,k} |b_{ik}|^2 \le \sum_{(i)} \int_0^1 |\varphi_i|^2 \, dx \cdot \sum_{(k)} \int_0^1 \left| \sum_{j=1}^{n-1} r_j \varphi_k^{(j)} \right|^3 dx;$$

computing the sums on the right side in the same manner as above, we have,

$$\sum_{i,k} |b_{ik}|^2 \le \int_0^1 \int_0^x \left| \frac{(x - t)^{n-1}}{(n - 1)!} \right|^2 \frac{dt}{p(t)} \frac{dx}{p(x)} \cdot$$

$$\int_0^1 \int_0^x \left| \sum_{j=0}^{n-1} r_j(x) \frac{(x - t)^{n-j-1}}{(n - j - 1)!} \right|^2 \frac{dt}{p(t)} \frac{dx}{p(x)},$$

which proves the boundedness of the sum of the series on the right-hand side.

This completely proves the above proposition on the convergence of B. G. Galerkin's method.

SECTION 2. THE SECOND ORDER EQUATION

We shall analyze in greater detail the case of the second-order equation. Let us consider the second-order equation

$$\frac{d}{dx}(py') + \frac{d}{dx}[q(x,\lambda)y] + r(x,\lambda)\, y = 0, \tag{21}$$

satisfying the conditions in Section 1, and let us consider boundary conditions of the type

$$\left.\begin{aligned} y'(0) &= h_0 y(0) \\ y'(1) &= h_1 y(1) \end{aligned}\right\}. \tag{22}$$

We shall prove the convergence of Galerkin's method under the following conditions imposed on the system (3):

a) The functions $\varphi_m(x)$ are twice differentiable, and satisfy the boundary conditions (22).

b) The set of derivatives $\varphi_m'(x)$ is complemented by the function $\psi(x)$ which is defined by the equalities

$$\psi(0) = h_0^{-1}, \qquad \psi(1) = -h_1^{-1},$$

$$\psi(x) = 1, \qquad 0 < x < 1,$$

if neither of the two numbers h_0 and h_1 is equal to zero and

$$\psi(x) = 0$$

for $h_0 = 0$ or $h_1 = 0$, and it is complete in the space, with distance

$$\rho(f,g) = \int_0^1 p|f - g|^2\, dx + \epsilon_0|f(0) - g(0)|^2 + \epsilon_1|f(1) - g(1)|^2,$$

where $\epsilon_i = 1$ if h_i is different from zero and infinity, and $\epsilon_1 = 0$ if $h_i = 0$ or $h_i = \infty$.

c) If $h_0 = h_1 = 0$, the set of function pairs (φ_j, φ_i) is complete in the space of pairs $\left(\int y\, dx,\ y\right)$ where y is a function with a summable square, and where distance is defined by the formula

$$\rho(f,g) = \int_0^1 (p|f - g|^2\, dx + |F\ (0) - G\ (0)|^2,$$

with

$$F\ (x) \int f\ dx, \quad G\ (x) = \int g\ dx.$$

In particular, condition c) is satisfied if the set of derivatives $\{\varphi_n'\ (x)\}$ is complete and the set of functions (3) contains unity.

We shall dwell on the case when h_0 and h_1 are different from zero and infinity. The remaining cases are considered analogously.

Let h_i be finite ($h_i \neq 0, \infty$). The set of derivatives of the functions of sequence (3) can be considered orthogonal in the space with distance ρ (f, g)

$$\int_0^1 p\varphi_i'\varphi_k' dx + \varphi_i'\ (0) + \varphi_k'\ (0) + \varphi_i'\ (1)\ \varphi_k'\ (1) = 0.$$

Let us consider the Fourier expansion of the function f with finite norm ρ (f, 0):

$$f \approx \sum c_i \varphi'_i. \tag{23}$$

This expansion converges in the space, with distance ρ (f, g). It follows from the definition of distance that this expansion converges in the mean on the interval $0 < x < 1$, and in the ordinary sense at the points $x = 0$ and $x = 1$.

By virtue of the boundary conditions (22) it can be easily shown that the functions $\varphi'_m(x)$ satisfy the relation

$$\int_0^1 \psi \, \varphi'_m \, dx + \psi(0) \, \varphi'_m(0) + \psi(1) \, \varphi'_m(1) = 0.$$

Keeping in mind that the set of functions

$$\psi, \ \varphi'_1, \ \varphi'_2, \ldots, \varphi'_m, \ldots$$

is complete in the space with distance

$$\rho(f,g) = \int_0^1 |f - g|^2 dx + |g(0) - f(0)|^2 + |g(1) - f(1)|^2,$$

We conclude that any function f satisfying the relations

$$\int_0^1 f\, \varphi_m' dx + f(0)\, \varphi_m' (0) + f\,(1)\, \varphi_m' (1) = 0,$$

is of the form $f = C\psi$, and that any function satisfying the relation

$$\int_0^1 f\, \psi\, dx + f\,(0)\, \psi\,(0) + f\,(1) \quad 1) = 0, \tag{24}$$

is an element of the linear space determined by the functional φ_m'. In particular, the Fourier series (23) converges to f in the space with distance $\rho\,(f, g)$.

The infinite system (10), which when truncated yields the equations for Galerkin's n-th approximation, can be obtained from the relations

$$\int_0^1 \left\{L(y) - f\right\} \varphi_i dx = \left[(py' + qy)\, \varphi_i\right]_0^1 -$$

$$-\int_0^1 [(py' + qy)\,\varphi'_i - (ry + f)\,\varphi_i]\,dx = 0 \qquad (25)$$

by formal substitution of the series

$$y = \sum x_i\varphi_i, \quad y' = \sum x_i\varphi'_i. \qquad (26)$$

If $y(x)$ is a solution of the system S, then y' satisfies relation (24) by virtue of the boundary conditions, and therefore the expansion converges to y' relative to the distance $\rho(f,g)$. The expansion for y converges uniformly by virtue of the convergence in the mean of the series for y' on $(0,1)$ and the convergence of the series

$$\sum x_i\varphi_i\ (0) = \frac{1}{h_0}\sum x_i\varphi'_i\ (0).$$

From the above there follows the validity of the formal substitution of the series (26) in (25), and therefore to the solution of system S, there corresponds the solution of equations (10) with convergent sum of squares

$$x_i = \int_0^1 py' \varphi'_i \, dx + y'(0)\, \varphi'_i(0) + y'(1)\, \varphi'_i(1).$$

Conversely, given the solution of system (10), with convergent sum of squares, we let

$$y = \sum x_i \varphi_i, \quad y_1 = \sum x_i \varphi'_i.$$

The second expansion converges relative to the distance $\rho(f,g)$ and the first uniformly. The function y is the integral of y_1; therefore, the relation obtained from (25) by replacing y' by y_1 is satisfied for the integrated terms (for $x = 0,1$). Integrating this relation by parts, and using the boundary conditions for φ_i, we obtain

$$\int_0^1 \left[py' + qy + \int_0^x (ry-f)dx \right] \varphi'_i dx + \frac{\varphi'(0)}{h_0} [py_1 \ (0) + qy\ (0)] -$$

$$- \frac{\varphi'_i(1)}{h_1} \left[py_1 + qy + \int_0^x [ry - f]\, dx \right]_{x=1} = 0,$$

whence

$$py' + qy + \int_0^x (ry - f)\, dx = C,$$

$$[py_1 + qy]_{x=0} = C,$$

$$[py_1 + qy]_{x=1} + \int_0^1 (ry - f)\, dx = C.$$

The first relation shows that y is a solution of the equation, and a comparison of the first relation with the other relations implies

$$y'(0) = y'(0), \quad y^1(1) = y_1(1),$$

because of the expansions of y and y_1 which converge for $x = 0,1$.

The function y satisfies the boundary conditions.

To complete the proof we must still establish the convergence of the series (19),

We have

$$A_{ik} = [p\varphi_k'\varphi_i + q\varphi_i\varphi_k - \varphi_k'\varphi_k]_{x=0} - [p\varphi_k'\varphi_i + q\varphi_i\varphi_k + \varphi_i'\varphi_k']_{x=0} +$$

$$+ \int_0^1 [q\varphi_i'\varphi_k - r\varphi_i\varphi_k]\, dx.$$

The proof of the convergence of the series is completely analogous to that given in section 1; it is only necessary to keep in mind the boundary conditions for the functions φ_i, and also the fact that an expression of the type

$$C\varphi_i'\ (0) + K\varphi_i' + (1) \int_0^1 f\varphi_i' dx$$

is the Fourier coefficient of a function which is equal to C for x = 0, K for x = 1, and f (x) for 0 < x < 1.

SECTION 3. THE DIRICHLET PROBLEM FOR EQUATIONS OF THE ELLIPTIC TYPE

Let us consider in the region D of the n-dimensional space

$x\ (x_1,\dots,x_n)$ the equation of the elliptic type

$$L(u) = \sum_{i,k=1}^{n} p_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{j=1}^{n} A_i \frac{\partial u}{\partial x_i} + Bu = f \quad (p_{ik} = p_{ki}), \qquad (27)$$

whose coefficients A_i, B depend linearly on the parameter λ, with the boundary condition

$$u = 0 \qquad (28)$$

on the boundary Γ of the region D.

The coefficients of equation (27) are assumed to be continuous with a sufficiently large number of partial derivatives with respect to the x_i, and the boundary of the region D is assumed to have continuous curvature. Because of the ellipticity of the equation, the quadratic form

$$\sum p_{ik}\, \xi_i \xi_k \qquad (29)$$

is positive, and we will assume that it is also nondegenerate on the boundary Γ of the region D.

Equation (27) will henceforth be written in the form

$$L(u) = \sum_{(i)} \frac{\partial}{\partial x_i}\left(\sum_k p_{ik} \frac{\partial u}{\partial x_k}\right) + \sum_{(i)} a_i \frac{\partial u}{\partial x_i} + bu = f, \tag{30}$$

$$a_i = a_i' + \lambda a_i'', \qquad b_i = b_i' + b_i''.$$

Let

$$\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots \tag{31}$$

be a set of functions having continuous second derivatives, and satisfying the boundary condition. As in the case of an ordinary equation, Galerkin's method consists in finding approximations to the solution of the problem in the form

$$u_m = \sum_{k=1}^{m} y_k^{(m)} \varphi_k, \tag{32}$$

with the constants $y_k^{(m)}$ determined from the system of equations

$$\int_{(D)} (L(u_m) - f)\, \varphi_j dx = 0. \tag{33}$$

We shall assume that the functions (31) satisfy the following condition C. Whatever the function $\Phi\ (x_1, \ldots, x_n)$, having an integrable square of the gradient

$$\int_{(D)} |\mathrm{grad}\ \Phi|^2\, dx < +\infty,$$

equal to zero on the boundary Γ, there exists a sequence of linear combinations Φ_m of the functions (31), the gradients of which converge in the mean to the gradient Φ,

$$\lim_{m\to\infty} \int |\mathrm{grad}\ (\Phi - \Phi_m)|^2\, dx = 0. \tag{34}$$

Then the eigenvalues of the equation (30) are obtained by a limiting process from the eigenvalues of system (33), and the gradients of the functions (32) converge in the mean to the gradient of the solution of equation (30).

We note that condition C is equivalent to the following requirement.

The set of vector fields consisting of

a) the gradients of the functions φ_m,

b) all vectors of the form

$$P_i = 0 \quad (i \neq j,\ k),\ P_j = \frac{\partial \varphi_m}{\partial x_k},\ \ P_k = -\frac{\partial \varphi_m}{\partial x_j},$$

c) the gradients of all harmonic polynomials, are complete in the space of vector fields $P(P_1, P_2, \ldots, P_n)$ with distance

$$\rho\,(P', P'') = \int\limits_{(D)} \sum_{i=1}^{n} |P_i' - P_i''|^2\, dx.$$

To prove the above proposition, we can assume that the gradients of the set of functions (31) are orthogonalized in the following sense.

$$\int\limits_{(D)} \sum_{i,k} p_{ik} \frac{\partial \varphi_m}{\partial x_i} \cdot \overline{\frac{\partial \varphi_j}{\partial x_k}}\, dx = \begin{cases} 1, & m = j, \\ 0, & m \neq j. \end{cases} \qquad (35)$$

For any vector P, we can construct Fourier coefficients from the formulas

$$y_m = \int_D \sum p_{ik} P_i \overline{\frac{\partial \varphi_m}{\partial x_k}} \, dx. \tag{36}$$

If the vector P is the gradient of a function which vanishes on the boundary D, or is approximated in the mean by the gradients of such functions, then equality

$$P = \sum_{j=1}^{\infty} y_j \operatorname{grad} \varphi_i. \tag{37}$$

is valid in the sense of convergence in the mean. A necessary and sufficient condition for convergence in the mean of the series (37) to some vector of the aforementioned type is the convergence of the series $\sum |y_j|^2$.

Along with the series (37) we shall also consider the series

$$\sum_{j=1}^{\infty} y_j \varphi_j. \tag{38}$$

From the convergence in the mean of the series $\sum' |y_j|^2$ follows the convergence in the mean of the series (38). In fact, denoting by ι_x the parallel segment of the axis x_1, connecting the point x with the contour Γ, and keeping in mind that φ_j vanishes on Γ, we have

$$\left|\sum y_j\varphi_j\right| = \left|\int_{\iota_x} \sum y_j \frac{\partial\varphi_j}{\partial x_1}\, dx_1\right| \leq \sqrt{\delta \cdot \int_{\iota_x} \left|\sum' y_j \frac{\partial\varphi_j}{\partial x_1}\right|^2 dx_1},$$

where δ is the diameter of the region D.

Therefore

$$\int_{(D)} \left|\sum' y_j\varphi_j\right|^2 dx \leq \delta \int_{(D)} \left|\sum' y_j \operatorname{grad} \varphi_j\right|^2 dx,$$

keeping in mind that

$$\sum_{i,k} p_{ik}\xi_i\xi_k > k \sum_{(i)}' \xi_i^2$$

and that by virtue of the orthogonality of the gradients φ_j

$$\int\limits_{(D)} \left| \sum y_j \varphi_j \right| dx \leq \frac{\partial}{k} \int\limits_{(D)} \sum_{i,k} p_{ik} \sum_{(i)} y_j \frac{\partial \varphi_j}{\partial x_i} \sum_{(g)} y_g \frac{\partial \varphi_g}{\partial x_i} dx <$$

$$< \frac{\delta}{k} \sum |y_j^2|. \tag{39}$$

From this inequality follows immediately the convergence in the mean of the series (38).

From (39) it also follows that from the convergence in the mean of the sequence

$$p^{(m)} = \sum y_j^{(m)} \text{ grad } \varphi_j \tag{40}$$

there ensues the convergence in the mean of the sequence

$$\varphi_m = \sum y_j \varphi_j. \tag{41}$$

We shall still need the following proportion: If the sequence (40) converges weakly to zero, then the sequence (41) also converges weakly to zero. Weak convergence of (40) to zero means that the norms

$$\int_{(D)} \sum_{(i)} |P_j^{(m)}|^2 \, dx$$

are bounded, and that for any vector **Q** with bounded norm

$$\lim_{m \to \infty} \int_D \sum_{(i)} Q_i P_i^{(m)} \, dx = 0.$$

From (39) follows the boundedness of the norms of the functions ψ_m; therefore to establish weak convergence of (41) it suffices to show that for any function g with integrable square

$$\lim_{m \to \infty} \int_D g\psi_m \, dx = 0. \tag{42}$$

Clearly it suffices to establish (42) for continuous functions g. Passing on to the limit in the equality

$$\int\limits_D g \cdot \sum y_j \varphi_j dx = \int\limits_D G \sum y_j \frac{\partial \varphi_j}{\partial x_1},$$

where

$$G = \int\limits_0^x g\ (x_1,\ x_2, \ldots, x_n)\ dx_1,$$

we obtain

$$\int\limits_D \psi_m g\ dx = \int\limits_D P_1^{(m)} G\ dx,$$

whence follows equality (42).

Let us consider the equation (43)

$$\Lambda\ (U) = \sum_{(i)} \frac{\partial}{\partial x_i} \sum_{(k)} p_{ik} \frac{\partial U}{\partial x_k}\ . \tag{43}$$

Because of the assumptions made about the coefficient p_{ik}, there exists a Green's function for equation (43), with the aid of which the solution of the equation is written in the form

$$U = \int_D G\ (x,\ \xi)\ F\ (\xi)\ d\xi. \tag{44}$$

Formula (44) gives the solution to the Dirichlet problem for equation (43), if the function $F\ (\xi)$ satisfies Hölder's condition

$$|F(\xi') - F(\xi'')| < k\ |\xi'\xi''|^{\lambda} \quad 0 < \lambda \leq 1,$$

where $|\xi'\xi''|$ is the distance between the points ξ', ξ''. In particular, (44) satisfies equation (43) if the function $F\ (\xi)$ has bounded partial derivatives.

It follows from the well-known properties of Green's function that this function itself and its partial derivatives can be written in the form

$$G\ (x,\ \xi) = \frac{\alpha(x,\ \xi)}{|x\xi|^{n-2}},$$

$$\frac{\partial G}{\partial x_1} = \frac{\alpha_i(x, \xi)}{|x\xi|^{n-1}},$$

where α, α_i are uniformly continuous functions of the variables x_i and ξ_k in the region D.

We shall need further the following proposition: If the function F (ξ) is summable in powers of $q \geq 1$, then the expression (44) and also

$$U_i = \int_D \frac{\partial G}{\partial x_i} F(\xi)\, d\xi \tag{45}$$

are defined almost everywhere, where the function $|U|^p$ is summable for $p \leq \frac{n}{n-2} q$, the function $|U_i|^p$ is summable for $p \leq \frac{n}{n-1} q$, and

$$\int_D |U|^p\, dx < C_1 \int_D |F|^q\, dx, \quad \int_D |U_i|^p\, dx \leq C_2 \int_D |F|^q\, dx.$$

This proposition follows immediately from the following lemma:

Lemma: Let $\alpha < n$, $q \geq 1$, $q \leq p < \frac{n}{\alpha} q$, let the function F (ξ) be defined and positive in the region D and $|F|^q$ be summable, and let Δ (x) be a subregion of D dependent on the point x, the

diameter of which is not larger than δ.
Then the function

$$I\ (x) = \int_{\Delta(x)} \frac{F(\xi)d\xi}{r^{\alpha}}, \quad r = |\ x\xi\ | \tag{46}$$

is finite almost everywhere in D, and

$$\int_D |\ I\ (x)\ |^p\ dx \le C\delta^{n\left(1 + \frac{1}{p} + \frac{1}{q}\right) - \alpha} \left(\int_D |\ F(\xi)\ |^q d\xi\right)^{\frac{1}{q}}, \tag{47}$$

where C is a constant depending on the numbers α, q, p, n.

Let us establish first the inequality (47) for the bounded function $F(\xi)$.

In this case $I(x)$ is finite everywhere. By virtue of a theorem of Riess(5) the function

$$G\ (\alpha, \beta) = \operatorname*{Max}_{F(\xi)} \frac{\int \left(\int_{\Delta(x)} \frac{|\ F(\xi)\ |}{r^{\alpha}}\ d\xi\right)^p dx}{\int_D |\ F\ (\xi)\ |^q\ d\xi}$$

is a logarithmically convex function of the variables $\alpha = \frac{1}{p}$, $\beta = \frac{1}{q}$ in the triangle $0 \le \alpha \le \beta$, $0 \le \beta \le 1$, i.e., for $0 < t < 1$, $\alpha = \alpha_1 t + \alpha_2 (1-t)$, $\beta = \beta_1 t + \beta_2 (1-t)$ we have

$$G(\alpha,\beta) \le G^{t}(\alpha_1, \beta_1) \cdot G^{(1-t)}(\alpha_2, \beta_2).$$

Computing the limits of the means for $\alpha = \beta = 0$, we have

$$G(0,0) = \operatorname*{Max}_{F(\xi)} \frac{\operatorname*{Max}_{x} \int_{\Delta(x)} \frac{F(\xi)\, d\xi}{r^{\alpha}}}{\operatorname*{Max}_{x} F(x)} = \operatorname*{Max}_{x} \int_{\Delta(x)} \frac{d\xi}{r^{\alpha}} \le C_1 \delta^{n-\alpha}$$

where C_1 depends only on n and α. On the other hand, because of Hölder's inequality, for $\alpha p < n$

$$\int_{\Delta(x)} \frac{F(\xi)\, d\xi}{r^{\alpha}} \le \left(\int_{\Delta(x)} \frac{F(\xi)\, d\xi}{r^{\alpha p}} \right)^{\frac{1}{p}} \left(\int_{\Delta(x)} F(\xi)\, d\xi \right)^{1-\frac{1}{p}},$$

Therefore

$$\left[\int_D \left(\int_{\Delta(x)} \frac{F(\xi)}{r^\alpha}\, d\xi\right)^p dx\right]^{\frac{1}{p}} \le \left(\int_D F(\xi)d\xi\right)^{1-\frac{1}{p}} \left[\int_D dx \int_{\Delta(x)} \frac{F(\xi)d\xi}{r^{\alpha p}}\right]^{\frac{1}{p}} \le$$

$$\le \int_D F\, d\xi \left[\operatorname*{Max}_x \int_D \frac{d\xi}{r^{\alpha p}}\right]^{\frac{1}{p}} = C_2 \delta^{\frac{n}{p} - \alpha} \int_D F\ (\xi)\ d\xi,$$

where C_2 depends only on n, p and α. The inequality obtained gives

$$G\left(\frac{1}{p},\ 1\right) \le C_2 \delta^{\frac{n}{p} - \alpha}.$$

Because of the logarithmic convexity of G, for $0 \le \frac{1}{q} \le 1$, $\frac{1}{q} \ge \frac{1}{p} \ge \frac{n}{\alpha q}$

$$G\left(\frac{1}{p}, \frac{1}{q}\right) \le G^{1-\frac{1}{q}}(0,0)G^{\frac{1}{q}}\left(\frac{q}{p}, 1\right) \le C_1^{1-\frac{1}{q}} C_2^{\frac{1}{q}} \delta^{n\left(1+\frac{1}{p}+\frac{1}{q}\right)-\alpha}.$$

The right-hand side of this inequality is finite, since $\alpha \frac{1}{q} < n$, and consequently (47) is proved for bounded $F(\xi)$.

Now let $F(\xi)$ be a function with summable q-th power. We will construct an increasing sequence of bounded functions $F_n(\xi)$ converging in the mean power q to $F(\xi)$.

By virtue of Lebesque's theorem

$$I_F(x) = \lim_{m\to\infty} J_{F_m}(x).$$

Since the sequence J^p_{Fm} increases, the inequality (47) for F is obtained by a limiting process from the corresponding inequality for F_m. It follows in particular, that the function $J_F(x)$ is finite almost everywhere.

Let us now turn our attention to the system of equations (33).

Keeping in mind the orthogonality of the functions φ_j (35), and in view of Green's theorem

$$\int_D u \sum_{(i)} \frac{\partial}{\partial x_i} \left(\sum_k p_{ik} \frac{\partial v}{\partial x_k} \right) dx = - \int_D p_{ik} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_k} dx,$$

which holds for any function u satisfying the relation (28), we can write the system of equations (33) in the form

$$y_j^{(m)} = - \sum_{s=1}^{m} A_{js} y_s^{(m)} + f_j$$

$$(j = 1,\ 2, \ldots, m), \tag{48}$$

where

$$\left. \begin{aligned} A_{js} &= \int_D \left(\sum_{(i)} a_i \frac{\partial \varphi_s}{\partial x_i} + b\varphi_s \right) \varphi_j \, dx, \\ f_j &= \int_D f \varphi_j \, dx. \end{aligned} \right\} \tag{49}$$

The system (39) is a truncation of the infinite system

$$y_j + \sum_{s=1}^{\infty} A_{js} y_s = f_j \quad (i = 1,\ 2, \ldots,). \tag{50}$$

Let us consider in Hilbert space the transformation

$$Y_j = f_j - \sum_{s=1}^{\infty} A_{js} y_s \tag{51}$$

We shall prove that for $\sum y_j^2 < +\infty$ and

$$u_i = \sum y_j \frac{\partial \varphi_j}{\partial x_i}, \quad u = \sum y_j \varphi_j \tag{52}$$

it follows from (51) that Y_j is the Fourier coefficient of the vector

$$U_j = - \int_D \frac{\partial G}{\partial x_j} \Big(\sum_{(i)} a_i u_i + bu - f \Big) dx. \tag{53}$$

In fact, putting

$$U_j^{(m)} = - \int_D \frac{\partial G}{\partial x_j} \Big(\sum_{(i)} a_i u_i^{(m)} + bu^{(m)} - f \Big) d\xi$$

$$u_i^{(m)} = \sum_1^m y_j \frac{\partial \varphi_j}{\partial x_i}, \quad u^{(m)} = \sum_1^m y_i \varphi_j$$

By virtue of Green's formula, we obtain

$$Y_j^{(m)} = \int_D \sum_{i,k} p_{ik} U_i^{(m)} \frac{\partial \varphi_j}{\partial x_k} d\xi = - \int_D \varphi_j A(U) \, d\xi =$$

$$= \int_D \varphi_j \Big(\sum a_i u_i^{(m)} + bu^{(m)} - f \Big) d\xi = - \sum_1^m A_{js} y_s + f_j.$$

By virtue of the lemma which has been proved, the convergence in the mean of $U_i^{(m)}$ to U_i follows from the convergence in the mean of the series (52), and consequently, $Y_j^{(m)}$ converges to the Fourier coefficient of the vector U_i. We shall note that in the sense of convergence in the mean

$$U_i = \sum Y_j \frac{\partial \varphi_j}{\partial x_i},$$

since U_i is approximated in the mean by the vectors $U_i^{(m)}$, which are the gradients of continuous functions which vanish on the boundary D.

We shall now prove that the Fourier coefficients of the gradient of the solution of equation (30) are a solution with a converging sum of squares of the system (50), and conversely, to any solution of system (50) with a converging sum of squares, there corresponds a solution of equation (30) whose gradient is determined by the formula

$$\text{grad } u = \sum y_i \ \text{grad } \varphi_j .$$

Let u be a solution of (30). Then in the sense of convergence in the mean

$$u = \sum y_j \varphi_j \ , \qquad \text{grad } u = \sum y_j \text{ grad } \varphi_j .$$

The solution u satisfies the relations

$$\int_D [L(u) - f] \varphi_j \, dx = 0,$$

and applying Green's formula we have

$$\int_D \Big[\sum_{i,k} p_{ik} \frac{\partial u}{\partial x_i} \frac{\partial \varphi_j}{\partial x_k} - \Big(\sum a_i \frac{\partial u}{\partial x_i} + bu - f \Big) \varphi_j \Big] dx = 0;$$

substituting in the above the expansions for u and grad u which converge in the mean, we find that Y_j satisfies the system of equations (50).

Conversely, if Y_j is a solution of system (50) with a converging sum of squares, then by virtue of the fact that the transformation (51) is equivalent to (53), and also putting

$$u_i = \sum y_j \frac{\partial \varphi_j}{\partial x}, \qquad u = \sum y_j \varphi_j,$$

we have

$$u_j = - \int_D \frac{\partial G}{\partial x_j} \Big(\sum a_i u_i + bu - f \Big) dx; \tag{54}$$

almost everywhere. We prove analogously that

$$u = - \int_D G \Big(\sum a_j u_j + bu - f \Big) dx \tag{55}$$

almost everywhere.

The functions $|u|^2$, $|u_i|^2$ are integrable, therefore $F = \sum a_i u_i + bu - f$ has also an integrable square. It follows from the above remark that both u and u_i are integrable in the power $p_1 = \dfrac{2n}{n-2}$. Applying repeatedly this proposition, we see that both u and u_i, and hence also the function F are integrable in the power $p_1 > \frac{n}{n-1}$.

But then, by virtue of Hölder's inequality

$$|u_j| \leq \left(\int_D \left|\frac{\partial G}{\partial x_j}\right|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_D |F|^p dx \right)^{\frac{1}{p}}$$

$$|u| \leq \left(\int_D |G|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_D |F|^p dx \right)^{\frac{1}{p}}.$$

Keeping in mind that $\frac{p(n - 1)}{p - 1} < n$, we conclude that u and the u_i are bounded. Now it follows from (54) that the u_i are continuous and satisfy the Hölder condition with any fractional index, and are equal to the derivatives of u. Because of this, F is continuous, and satisfies Hölder's condition, and therefore u satisfies equation (30) and the boundary condition (28).

From the equivalence of the Dirichlet problem for equation (30) and the system (50), which has been proved, there follows, in particular, the coincidence of the eigenvalues for system (50) and equation (30).

We shall now prove that the eigenvalues of system (50) and its solution are obtained by a limiting process from system (48) where

$$\lim_{m\to\infty} \sum_{j=1}^{m} |y_i - y_j^{(m)}|^2 = 0.$$

From this will clearly follow the above-mentioned proposition about Galerkin's method.

To establish these properties for system (50) it suffices to show that the transformation (51) is completely continuous*, i.e., that weak convergence to zero of $y(y_1, y_2, \ldots, y_m, \ldots)$

$$\lim_{m\to\infty} y_j^{(m)} = 0, \qquad \sum_{j=1}^{\infty} |y_j^{(m)}|^2 < M$$

implies strong convergence to zero of the sequence $Y^{(m)}$

---

*It can be easily seen that in the case of partial differential equations the series $\sum |A_{js}|^2$ generally no longer converges. To this end it suffices to consider the equation

$$\Delta\varphi + \lambda \frac{\partial\varphi}{\partial x} = 0$$

on the square $0 < x,\ y < \pi$ and the set of approximating functions

$$\varphi_{m,n} = \frac{4}{\pi^2} \frac{\sin nx \cdot \sin mx}{n^2 + m^2}.$$

$$\lim_{m\to\infty} \sum_{j=1}^{\infty} |Y_j^{(m)}|^2 = 0.$$

The numbers $Y_j^{(m)}$ are defined as the Fourier coefficients of the vector $U_j^{(m)}$ which is determined by the formula (53); therefore it suffices to show that from weak convergence to zero of the functions $F_m(\xi)$

$$\lim_{m\to\infty} \int_D F_m\ (\xi)\ g\ (\xi)\ d\xi = 0, \qquad \int_D |F_m\ (\xi)|^2 d\xi < M, \tag{56}$$

there follows convergence in the mean to zero of the vectors

$$U_j^{(m)} = \int_D \frac{\partial G}{\partial x_j} F_m\ (\xi)\ d\xi.$$

To see this, we make first of all the following remark.

If the function $H\ (x,\ \xi)$ is uniformly continuous when x and ξ vary in D, then from (56) follows uniform convergence to zero of the function

$$h_m(x) = \int_D H(x, \xi) F_m(\xi) d\xi.$$

In fact, for an arbitrary E the region D can be decomposed into a finite number of parts D_α, in such a way that the point ξ which lies in D_α satisfies the inequality

$$|H(x, \xi) - H_\alpha(x)| < E, \quad H_\alpha = H(x, \xi_\alpha),$$

where ξ_α is a fixed point of D_α. Then

$$|h_m(x)| \leq E \int_D |F_m| dx + \sum_{(\alpha)} |H_\alpha(x)| \cdot \left| \int_{D_\alpha} F_m d \right| \leq E \sqrt{M \cdot \text{volume } D} +$$

$$+ \max |H(x, \xi)| \cdot \sum_{(\alpha)} \left| \int_{D_\alpha} F_m d\xi \right.$$

and since D_α is independent of x, this shows uniform convergence of h_m to zero.

Because of the above-mentioned property of Green's function $G(x, \xi)$, we can represent its derivatives in the form

$$\frac{\partial G}{\partial x_j} = \alpha_j (x, \xi) + \beta_j (x, \xi),$$

where the terms $\beta_j (x, \xi)$ are uniformly continuous as x varies in the region D, and the first terms satisfy the inequalities

$$|\alpha_j (x, \xi)| < \frac{A}{|x\xi|^{n-1}}$$

and vanish for

$$|x\xi| > \delta,$$

where δ is arbitrarily small.

Then

$$U_j^{(m)} = \int_D \alpha_j (x, \xi) F_m (\xi) d\xi + \int_D \beta_j (x, \xi) F_m (\xi) d\xi.$$

The first terms of these expressions $\overline{U}^{(m)}(x)$ satisfy, by virtue of our lemma the inequalities

$$\left[\int_D |\overline{U}_j^{(m)}|^2 dx \right]^{\frac{1}{2}} < CA\delta \left[\int_D |F|^2 d\xi \right]^{\frac{1}{2}}$$

and consequently,

$$\int_D |\overline{U}_j^{(m)}|^2 \, dx \leq (CA\delta)^2 \, M.$$

The second terms, because of weak convergence to zero of the sequence $F_m(\xi)$ and uniform continuity of the kernels $\beta_j\ (x,\ \xi)$, converge uniformly to zero. Since δ can be chosen arbitrarily small, this proves the strong convergence of $U_j^{(m)}$ to zero.

Received 31 March 1942

FARADAY TRANSLATIONS
15 PARK ROW
NEW YORK 38, N. Y.